Conclusion

By using the complex modal analysis,[1-4] Caughey's normal mode approach to nonlinear random vibration problems has been successfully generalized in two respects. First, the random excitation may be white or colored noise, and no restrictions will be imposed on the correlations among the random excitations. Second, the linear part of the nonlinear system may be an arbitrary linear damped system. The generalization expands the usefulness of the normal mode approach and retains the previous computational advantages. The generalization does not address the limitation on the nonlinearity per se, which remains the same as in Caughey's approach.

Appendix

By using an equivalent linear element in place of the system nonlinearity, an equation error, e_i, is obtained as

$$e_i = (\hat{p}_i - p_i)z_i + n_i(z)$$

Letting

$$\hat{p}_i - p_i = a + jb$$

we have

$$\langle e_i \bar{e}_i \rangle = (a^2 + b^2)\langle z_i \bar{z}_i \rangle + 2a\mathrm{Re}\langle n_i \bar{z}_i \rangle + 2b\mathrm{Im}\langle n_i \bar{z}_i \rangle + \langle n_i \bar{n}_i \rangle$$

The parameters a and b may be determined by rendering the mean square value $\langle e_i \bar{e}_i \rangle$ to be a minimum, i.e.,

$$\langle e_i \bar{e}_i \rangle = \min \tag{A1}$$

The necessary (and also sufficient in this case) conditions for Eq. (A1) to be true are

$$(\partial/\partial a)\langle e_i \bar{e}_i \rangle = 0, \quad (\partial/\partial b)\langle e_i \bar{e}_i \rangle = 0$$

These conditions lead to

$$a\langle z_i \bar{z}_i \rangle + \mathrm{Re}\langle n_i \bar{z}_i \rangle = 0$$

$$b\langle z_i \bar{z}_i \rangle + \mathrm{Im}\langle n_i \bar{z}_i \rangle = 0 \tag{A2}$$

Combining the last two equations into a single complex one, we obtain

$$(\hat{p}_i - p_i)\langle z_i \bar{z}_i \rangle + \langle n_i \bar{z}_i \rangle = 0$$

Hence

$$\hat{p}_i = p_i - \langle n_i \bar{z}_i \rangle / \langle z_i \bar{z}_i \rangle$$

Because

$$(\partial^2/\partial a^2)\langle e_i \bar{e}_i \rangle = 2\langle z_i \bar{z}_i \rangle > 0$$

$$(\partial^2/\partial b^2)\langle e_i \bar{e}_i \rangle = 2\langle n_i \bar{n}_i \rangle > 0$$

$$(\partial^2/\partial a \partial b)\langle e_i \bar{e}_i \rangle = 0$$

$\hat{p}_i$ makes $\langle e_i \bar{e}_i \rangle$ a minimum.

Acknowledgment

The authors are grateful to Prof. E. H. Dowell for his valuable comments.

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Buckling of Irregular Plates by Splined Finite Strips

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Introduction

THIS Note presents the results of an investigation into an alternative finite strip method developed using cubic x-spline functions. The buckling of constant-thickness plates with rectangular shapes has been investigated by many authors. The assumptions made are those of the classical plate theory as described by Timoshenko and Gere.[1] The stability of plates using the finite element method was developed by Kapur and Hartz,[2] Anderson et al.,[3] and others. The finite strip method, with polynomials in the x direction and continuous differentiable smooth (trigonometric) series in the y direction,[4,5] is a powerful method for problems defined in rectangular domains, including plates with stiffeners. The modified finite strip method was developed for the flexural analysis of irregular plates, using spline functions instead of trigonometric series.[6] With this formulation, rectangular, nonrectangular, irregular-shaped, two-dimensional problems can be investigated. Buckling analysis of nonrectangular plates, such as trapezoidal and polygonal plates, is presented.

Theory of Buckling

The finite element theory of elastic stability of plates has been developed by Anderson et al.[3] The algebraic equation of the stability problem of a plate is as follows:

$$([K] + \lambda[K^G])\{\delta\} = \{0\} \tag{1}$$

This is an eigenvalue problem similar to the vibration problem.[7] It is equivalent to the determination of the eigenvalue of the following determinant as indicated by Cheung et al.[8]

Received Feb. 5, 1985; presented as Paper 85-0637 at the AIAA/ASME/ASCE/AHS 26th Structures, Structural Dynamics and Materials Conference, Orlando, FL, April 15-17, 1985; revision received July 4, 1985.

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$$|[K]+\lambda_b[K^G]|=0 \tag{2}$$

in which λ_b is a buckling factor related to the critical load, $[K]$ a global stiffness matrix, and $[K^G]$ a global geometric stiffness matrix for the whole plate.

Spline Functions in Finite Strips of Plates

The cubic x-spline functions were developed by Clenshaw and Negus.[9] These functions can be applied as interpolating functions in the y direction in the finite strip analysis of the plate-bending problems. Using the cubic x-spline function as the Hermite interpolating formula gives the function $\bar{w}(\bar{y})$, which may be represented as

$$\bar{w}(\bar{y})=h_jf_{1j}\theta_j-h_jf_{2j}\theta_{j+1}+f_{3j}w_j+f_{4j}w_{j+1} \tag{3}$$

where $\bar{w}(\bar{y})$ is determined in terms of w_j, θ_j, w_{j+1}, and θ_{j+1}, and its derivative $[\mathrm{d}\bar{w}(\bar{y})]/\mathrm{d}y$ at knot j in the interval, $j=r$, $r+1$, ..., k, $k+1$. h_j is the length of the jth section, i.e.,

$$h_j=y_{j+1}-y_j \tag{4}$$

f_{1j}, f_{2j}, f_{3j}, and f_{4j} are cubic interpolating functions.

The equations defining the cubic x spline are

$$\beta_{j-1}\theta_{j-1}+\theta_j=(1-\beta_{j-1})^2\bar{W}_j+\beta_{j-1}(3-\beta_{j-1})\bar{W}_{j-1}$$

$$j=r+1,r+2,\ldots,k \tag{5}$$

where

$$\beta_j=\frac{h_{j+1}}{h_j+h_{j+1}};\qquad \bar{W}_j=\frac{w_{j+1}-w_j}{h_j} \tag{6}$$

Equation (5) can be solved for θ_j *as*

$$\theta_j=U_{j-1}-\beta_{j-1}[U_{j-2}-\beta_{j-2}[U_{j-3}$$

$$-\beta_{j-3}[U_{j-4}-\ldots\beta_{r+1}(U_r-\beta_r\theta_r)]]] \tag{7}$$

where

$$U_j=(1-\beta_j)^2\bar{W}_{j+1}+\beta_{j+1}(3-\beta_j)\bar{W}_j \tag{8}$$

θ_r and θ_{k+1} are end parameters. Substitution of Eq. (7) into Eq. (3) yields the result

$$\bar{w}(\bar{y})=[F]_j\{w\}_j \tag{9}$$

where $[F]_j$ are row matrix functions of the cubic interpolating functions and $\{w\}_j$ are column matrices of the knot parameters of the jth section of a strip. The deflection function $w(\bar{x},\bar{y})$ is an element of the ith strip of the plate expressed by a polynomial interpolating function in the x direction, i.e.,

$$w(\bar{x},\bar{y})=\phi_{1i}\bar{w}_i(\bar{y})+b_i\phi_{2i}\bar{w}_i'(\bar{y})+\phi_{3i}\bar{w}_{i+1}(\bar{y})$$

$$+b_i\phi_{4i}\bar{w}_{i+1}'(\bar{y}) \tag{10}$$

ϕ_{1i}, ϕ_{2i}, ϕ_{3i}, and ϕ_{4i} are conventional cubic interpolating functions in the x direction and $\bar{w}_i'(\bar{y})$ and $\bar{w}_{i+1}'(\bar{y})$ *are* $\bar{w}_i(\bar{y})$ and $\bar{w}_{i+1}(\bar{y})$ derivatives, respectively, of the ith strip. Substitution of Eq. (9) into Eq. (10) yields

$$w(\bar{x},\bar{y})=[N]_j\{\delta\}_j \tag{11}$$

where

$$[N]_j=[[G_i]_j[G_{i+1}]_j] \tag{12}$$

$$\{\delta\}_j=\begin{Bmatrix}\{\delta_i\}_j\\ \{\delta_{i+1}\}_j\end{Bmatrix} \tag{13}$$

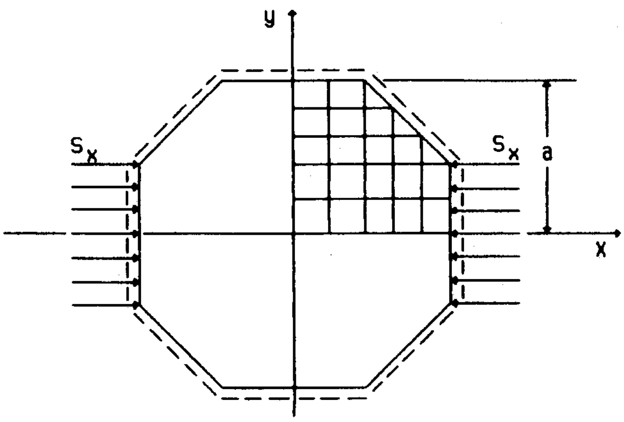

Fig. 1 Simply supported equilateral octagonal plate (with 5 strips × 5 elements on one-quarter of octagon).

$[G_i]_j$ and $[G_{i+1}]$ j are row matrix functions of geometry and cubic interpolating functions[6,12]; $\{\delta_i\}_j$ and $\{\delta_{i+1}\}_j$ are column matrices for corresponding nodal parameters along the ith strip.

Algorithm for Large Eigenvalue Problems

A method has been presented for the solution of the eigenvalue problem[10,12] of the form of Eq. (2). The element stiffness and stability (geometric stiffness) matrices have been derived in Ref. 12.

In Eq. (2), $[K]$ and $[K^G]$ are both real and symmetrical matrices.[11] The eigenvalue problem of determining the nontrivial solutions by numerical computations[13] is used. A computer program based on the QZ algorithm[10] has been used to find all of the eigenvalues and their corresponding eigenvectors. The critical load corresponds to the lowest eigenvalue, the lowest buckling factor λ_b, is related to the fundamental critical load.

Numerical Examples

1) The results of a convergence test of the proposed method on buckling factors in uniformly compression loading of a simply supported square plate is shown in Table 1. The buckling factor λ_b is defined as

$$\lambda_b=\frac{12(1-\nu^2)}{\pi^2E}\left(\frac{a}{t}\right)^2\sigma_{\mathrm{cr}} \tag{14}$$

where a is the width of the plate and σ_{cr} the critical stress. From Table 1, it can be seen that even a 3 strip × 3 element model gives satisfactory results and that the convergence is good.

Table 1 Convergence test for a square plate subjected to compression in one direction

Strips × elements	2×2	3×3	4×4	5×5	Ref. 1
Buckling factors	3.9440	3.9855	3.9970	3.9995	4.0000

Table 2 Trapezoidal plate subjected to uniform compression in the x direction

Strips × elements	2×6	2×8	3×9	3×11	4×11
Buckling factors	4.5750	4.9076	5.6493	6.1063	6.1276

Table 3 Equilateral octagonal plates subjected to uniform compression in the x direction

Strips × elements	4×4	5×4	5×5	5×6	6×5
Buckling factors	11.524	6.397	4.534	4.532	4.526

Table 4 Twelve-sided plates with uniform compression in the x direction

Strips × elements	4×4	4×5	5×4	5×5	5×6
Buckling factors	5.579	4.907	4.677	4.348	4.334

Table 5 Sixteen-sided plates subjected to uniform compression in the x direction

Strips × elements	4×4	4×5	5×5	5×6	6×6
Buckling factors	5.297	4.605	4.292	4.277	4.272

Table 6 Buckling factors of equilateral polygons

No. of sides of polygon	8	12	16	20
Buckling factors	4.526	4.334	4.272	4.208

2) For a simply supported trapezoidal plate under uniformly compressive loading in the x-direction, the finest division used is a 4 strip × 11 element model on one-half of the trapezoidal plate. The plate is isosceles with a height of (a), the parallel sides measure (a and $3a$), and the height/thickness ratio (a/t) is 100. The average width of ($2a$) is used in lieu of (a) in Eq. (14). It is subjected to uniform normal compression along the parallel sides. Buckling factors (with increasing strips and elements) are shown in Table 2. The buckling factor λ_b approaches 6.1.

3)The buckling factors shown in Table 3 are for simply supported equilateral octagonal plates under compressive loading in the x direction (Fig. 1).

4) The results of a simply supported, equilateral, 12-sided plate are investigated. The results are shown in Table 4.

5) The buckling factors of a simply supported, equilateral, 16-sided polygon plate are shown in Table 5.

6) Table 6 lists the buckling factors for different equilateral polygons, including the circle (a polygon approximated by 20 sides). It can be seen that the buckling factor of equilateral polygons approaches that of a circle as the number of sides increase.

Summary

It can be seen from illustrative examples and tables that the proposed method converges rapidly and accurately when compared with known solutions. The computational labor for the proposed method is about 2 to 3 times less than that used by finite element programs of similar accuracy.[6] Based on the authors' experience, more elements (compared with number of strips) give better accuracy.

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Vibrations of Infinitely Long Cylindrical Shells of Noncircular Cross Section

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Displacement Equations of Motion

CONSIDER a thin-walled, closed, infinitely long cylindrical shell having a noncircular cross section of constant thickness h and made of an isotropic, linearly elastic material. The shell is referred to a right-hand system of orthogonal curvilinear coordinates x^*, s^*, and z. x^* is measured along the axis of the shell, s^* along the curve formed by the intersection of the plane normal to the axis of the shell and its middle surface, and z inward along the direction perpendicular to the middle surface of the shell.

The x^*, s^* coordinates and the radius of curvature r^* of the cross section of the shell are nondimensionalized with

Received March 4, 1985; revision received May 15, 1985.

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